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3D Graphic Statics: Geometric Construction of Global Equilibrium

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Abstract

Graphic Statics provides a collection of procedures for the design and analysis of two-dimensional structural systems, involving only geometric operations on two-dimensional form and force diagrams. Unfortunately, the applicability of Graphic Statics to three-dimensional problems is limited. Only three-dimensional problems that can be replaced by an equivalent series of two-dimensional problems, can be addressed.

It has been shown how the equilibrium of three-dimensional systems of forces can be described using polyhedral form and force diagrams [1, 2, 7]. However, procedures for solving specific design and analysis problems with these polyhedral diagrams have not yet been developed.

One of the fundamental procedures in traditional (2D) Graphic Statics is to establish global equilibrium for given boundary conditions, and to use this information to construct different funicular solutions for the specified loads and supports. Therefore, in this paper, as a first step in the development of truly 3D Graphic Statics, we describe an equivalent procedure for given, three-dimensional boundary conditions. The procedure involves only geometric operations on polyhedral form and force diagrams.

The method, as it is presented here, is only applicable to statically determined systems of forces in which the applied loads do not generate a resultant couple. We give an overview of the well known procedure in 2D Graphic Statics to identify its key concepts and constructive elements, and describe the different steps of the three-dimensional version in detail.

Keywords: 3D graphic statics, reciprocal form and force polyhedrons, global force polyhedron, funicular polyhedron, 2D graphic statics, global force polygon, funicular polygon.

1. Introduction

Graphic Statics could be defined as a collection of procedures involving only geometric construction techniques (applied to so-called form and force diagrams) for design and analysis of two-dimensional structures. The applicability of these procedures to three-dimensional problems is limited to those problems that can be simplified to an equivalent series of interrelated two-dimensional problems.

In "The Principle of Equilibrium of Polyhedral Frames", Rankine [7] proposed how the equilibrium of a spatial system of forces, applied to a single point in space, could be described using a closed polyhedron. Based on Rankine's proposition, Akbarzadeh *et al.* [1, 2] visualized and clarified the reciprocal relationship between polyhedral form and force diagrams and demonstrated their potential for the design of complex spatial structures. For example, they showed how the aggregation of convex polyhedral force cells could be used to specifically generate complex spatial systems of forces in compression-only (or tension-only) equilibrium (Akbarzadeh *et al.* [1 2]). However, as the described procedures did not allow for the generation of specific solutions for given boundary conditions, such as specific support locations and/or specific applied loads, their applicability in actual design situations remains limited. Therefore, it is clear that to go beyond open-ended explorations of three-dimensional force equilibrium, and develop an actual three-dimensional version of Graphic Statics, procedures should be established through which specific structural problems involving spatial systems of forces can be addressed in a rigorous and intuitive manner, similar to the way in which specific two-dimensional problems can be addressed with the specific procedures offered by traditional (2D) Graphic Statics.

In this paper we describe a method for exploring global equilibrium of different funicular solutions for given three-dimensional boundary conditions, such as the spatial configuration of loads and supports, based only on geometric operations involving form and force polyhedrons. Note that the presented procedure is only valid if the applied loads do not generate a resultant couple, i.e. if they can be replaced by a resultant force alone. The procedure is furthermore only applicable to (externally) statically determined structures.

In Section 2, we give an overview of the corresponding procedure in 2D Graphic Statics, to identify the key steps and constructive elements for which we should provide a three-dimensional equivalent. This includes 1) determining the magnitude, direction and location of the resultant force, 2) establishing the requirements for global equilibrium for given boundary conditions, and 3) constructing funicular solutions for specified support locations. In Section 3, we give a detailed description of the 3D method.

2. 2D Graphic Statics

The 2D Graphic Statics procedure for constructing funicular solutions for given boundary conditions, i.e. given loads and support locations, is well known and well documented (Allen and Zalewski [3], Bow [4], Cremona [5], Ritter [8] and Wolfe [9]). Therefore, we only give a brief overview here, to identify the key aspects and constructive elements for which we will provide three-dimensional equivalents in Section 3.

2.1. Resultant force

Figure 1a shows how the magnitude and direction of the resultant force of the given loads are found by constructing a load line, and how the location of the line of action of the resultant force in relation to the applied loads is determined using a trial construction. The trial construction involves a decomposition of the forces in the load line using an arbitrary trial pole, P_{trial}, and the construction of a corresponding trial funicular in the form diagram. See Wolfe [9] for a detailed explanation of this procedure.

Figure 1b depicts an alternative procedure for finding the location of the line of action of the resultant force. This procedure has a direct three-dimensional equivalent, as we will see in Section 3. The procedure consists of the following steps. First, we construct a line perpendicular to the line of action of the resultant force in the load line. We will call this the equilibrium line. Then, we intersect each of the lines of action of the loads with the equilibrium line, and decompose the loads at these intersection points into two components, one parallel to the equilibrium line and one perpendicular. Note that the perpendicular components are thus parallel to the resultant force. The intersection of the line of action of the resultant and the equilibrium line is the centroid of the intersection points of the applied loads, weighted by their perpendicular components. Figure 1b shows the geometric construction of this weighted centroid. The construction is basically the repeated application of a graphical method for finding the resultant of two parallel forces, described in detail in Wolfe [9].

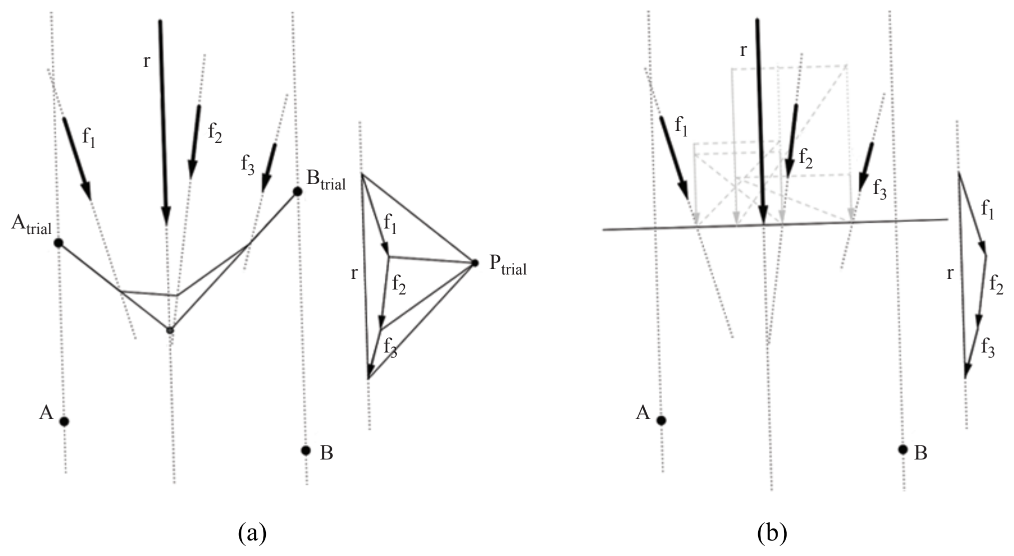

Figure 1: a) The location of the line of action of the resultant can be determined using the well-known trial construction. b) An alternative approach is the construction of the weighted centroid of the intersections of the lines of action of the applied loads with a line perpendicular to the line of action of the resultant in the load line. The intersections are weighted by the magnitude of the perpendicular components of the loads with respect to this line.

2.2. Global equilibrium

Having determined the location of the line of action of the resultant force, we can use a trial construction to determine the requirements for global equilibrium for the given boundary conditions. Essential to this is the trial closing string (Allen and Zalewski [3] and Wolfe [9]).

Figure 2a depicts this trial construction. It is the result of the following steps. First, we draw lines through the support locations A and B, parallel to the line of action of the resultant force. As before, we choose an arbitrary pole in the force diagram, P_{trial}, to decompose the resultant force. Then, from an arbitrary point A_{trial} on the line through A, we construct a line parallel to 1-P_{trial}. This line intersects the line of action of the resultant force in R_{trial}. From R_{trial} we then construct a line parallel to 2-P_{trial}, which intersects the line through B in B_{trial}. The trial closing string is the line connecting points A_{trial} and B_{trial}.

The line through the trial pole, parallel to A_{trial}-B_{trial}, intersects the resultant force in the load line at point X. This point divides the resultant force into two components, a and b (see Figure 2a). These components, when applied anywhere on the lines through A and B, create force and moment equilibrium for the given loads. In Figure 2a, we have applied the resultant force and the two reaction components at points on the equilibrium line. Note that equilibrium could be verified using the geometric procedure for finding the resultant of two parallel forces.

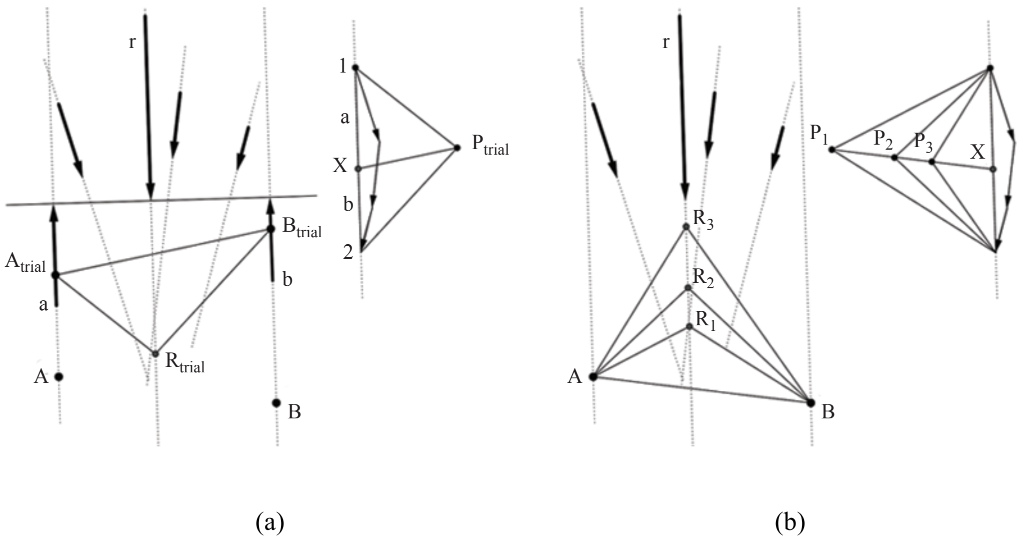

Figure 2: a) The trial closing string can be used to determine the conditions for equilibrium for the given boundary conditions. b) Based on this, all funicular solutions through the prescribed support locations can be constructed.

2.3. Funicular solutions

Figure 2b shows how the information about global equilibrium for the given boundary conditions, represented by point X, can be used to generate infinitely many funicular solutions through the specified support locations.

First, we construct a new closing string connecting the prescribed supports A and B, and a line parallel to this closing string through X. Any pole on this line, generates a different funicular solution for the given boundary conditions. Figure 2b depicts three poles, P_1, P_2 and P_3, with three corresponding funicular solutions, R_1, R_2 and R_3.

3. 3D Graphic Statics

In this section, we describe a geometric procedure for generating funicular solutions for given boundary conditions involving spatial configurations of applied loads and supports. As in the two-dimensional case, the procedure consists of three main steps: 1) determine the magnitude, direction and location of the resultant force (and couple), 2) determine the requirements for global equilibrium for the given boundary conditions, and 3) construct funicular solutions through the specified supports.

Unlike in two dimensions, a three-dimensional system of forces cannot always be replaced by a resultant force alone. In some cases, a resultant couple is produced as well. The second and third step of the presented procedure are only valid if the loads do not have a resultant couple. The procedure is furthermore only valid for statically determined systems.

3.1. Resultant force and couple

Consider the spatial configuration of forces depicted in Figure 3. As in the two-dimensional case, we find the magnitude and direction of the resultant force by constructing a load line. Then, we use a similar procedure to the alternative procedure described in Section 2.2 in relation to Figure 1b, to determine the location of the line of action of the resultant.

First, we construct a plane perpendicular to the line of action of the resultant force. This is the equilibrium plane, which is the three-dimensional equivalent of the equilibrium line introduced in Section 2.1. Then, we find the intersection points of the lines of action of the loads with the equilibrium plane, and decompose the loads at these intersection points into an in-plane and a normal component. Note that the normal components are parallel to the resultant force. As in Section 2.2, the intersection of the line of action of the resultant force with the equilibrium plane is the centroid of the intersections of the loads with the resultant plane, weighted by the magnitude of their normal components.

The resultant force of the in-plane components is zero. However, depending on the configuration of the loads, the in-plane components can produce a resultant couple around an axis normal to the equilibrium plane (Figure 3b). The geometric construction method presented in this paper is only valid for configurations of loads that do not generate a resultant couple, and can thus be replaced by a resultant force alone (Figure 3a). This is, for example, always the case if the loads are concurrent, i.e. if their lines of action intersect at a single point in space. It is also the case if the loads are parallel.

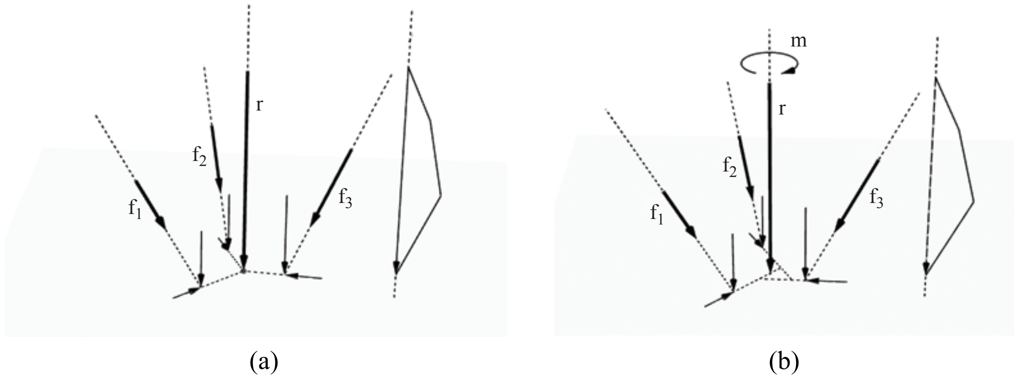

Figure 3: The magnitude and direction of the resultant force of a spatial system of forces can be found by constructing a load line. The location of the line of action of the resultant force is the centroid of the weighted intersection points of the lines of action of the loads with the equilibrium plane. The weights on the intersection points are the magnitudes of the components of the loads perpendicular to the equilibrium plane. a) The forces do not produce a resultant couple. Therefore, they can be replaced by a resultant force alone. b) The forces produce a resultant force and a resultant couple.

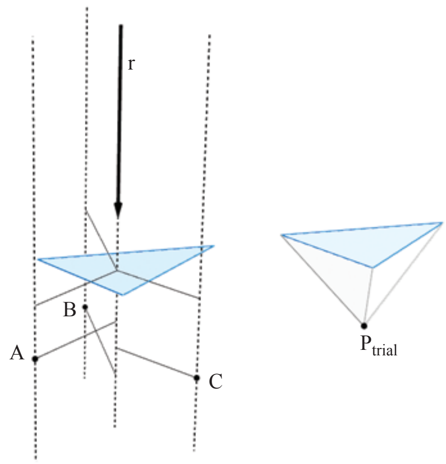

Figure 4: In the force polyhedron, the resultant face is a triangle if the structural system is (externally) statically determined. The area of the triangle is equal/proportional to the magnitude of the resultant force, and its sides perpendicular to the lines of action of the reaction forces.

3.2. Global equilibrium

The next step is to use a trial construction to establish the requirements for global equilibrium for the given boundary conditions. First, we construct a trial force polyhedron. The global force polyhedron of a statically determined system of forces is a tetrahedron. Therefore, the face corresponding to the resultant force is a triangle, perpendicular to the line of action of the resultant force. The area of the triangle is equal (or proportionate) to the magnitude of the resultant force. The sides of the triangle are perpendicular to the lines of action of the corresponding reaction forces. If the applied loads have no resultant couple, the lines of action of the applied loads intersect in a point on the line of action of the resultant force. Using a trial pole, we then complete the trial force polyhedron by connecting the pole to the resultant face. The construction of the resultant face and the trial force polyhedron are depicted in Figure 4.

In the form diagram, we then construct the trial funicular as follows. First, we draw lines through the prescribed support locations, A, B and C, parallel to the line of action of the resultant. From an arbitrary point A_{trial} on the line through A we construct a line perpendicular to the corresponding face in the force polyhedron. This line intersects the line of action of the resultant force at point R_{trial} (Figure 5). From R_{trial}, we construct lines perpendicular to the other two reaction faces of the trial force polyhedron to find intersections B_{trial} and C_{trial} (Figure 5) on the lines through the other two support locations.

The plane defined by the points A_{trial}, B_{trial} and C_{trial} is the three-dimensional equivalent of the trial closing string in 2D. Drawing a line perpendicular to this plane, through the trial pole in the force polyhedron, we find point X (Figure 5). As in the two-dimensional construction, this point divides the resultant face into three components, and thereby defines the required distribution of reaction forces along the lines parallel to the line of action of the resultant through the support locations to establish global equilibrium for the given boundary conditions.

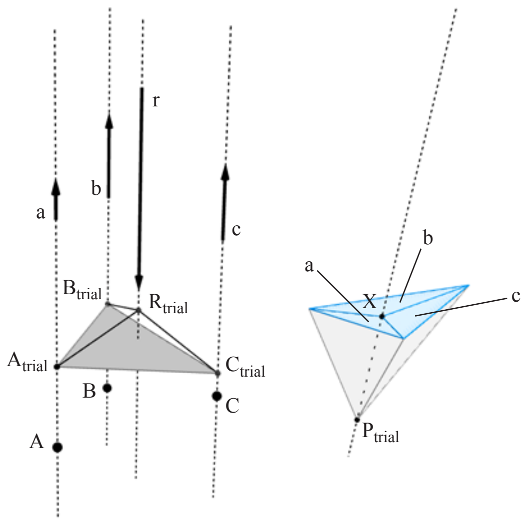

Figure 5: A line through the trial pole, perpendicular to the closing plane of the trial construction, intersects the resultant face in X.

3.3. Funicular solutions

Having determined the location of X, we can now easily construct funicular solutions for the specified support locations. First, we construct a closing plane through A, B and C, and a line perpendicular to this plane, through X (Figure 6). As in the two-dimensional case, any pole point on this line, forming a closed force polyhedron together with the resultant face, represents a funicular solution for the given boundary conditions. Several solutions, sharing the same supports, are depicted in Figure 6.

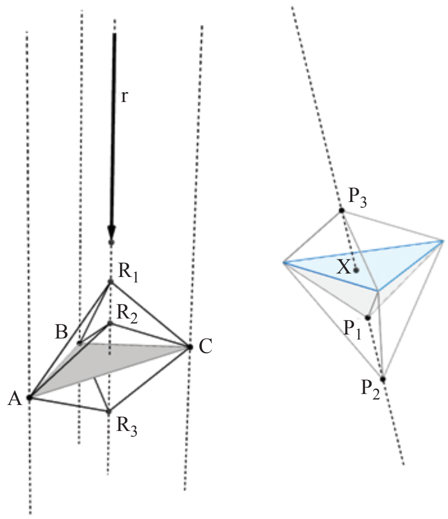

Figure 6: Every pole on a line through X, perpendicular to the closing plane of the specified supports (A, B, C), defines a different funicular solution for the given boundary conditions.

Conclusion

In this paper, we have presented a purely geometric method for exploring global equilibrium of different funicular solutions for given three-dimensional boundary conditions, using polyhedral form and force diagrams. The method is the three-dimensional version of the equivalent, well known procedure of traditional (2D) Graphic Statics.

The method, as it is presented here, can only be applied to statically determined systems of forces with applied loads that do not produce a resultant couple. In other words, the method can only be used if the applied loads can be replaced by a resultant force alone.

Future work should focus on extending this method to general systems of forces (including systems of which the applied loads generate a resultant couple), and on the development of procedures for translating the obtained results for global equilibrium to internal force distributions.

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