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Graphic Statics: Constrained form finding for parallel system of forces using Corsican sum

Georgios-Spyridon ATHANASOPOULOS*, Masoud AKBARZADEH[a] , Allan MCROBIE[b]

*,[b] University of Cambridge, Department of Engineering
Trumpington Street, CB2 1PZ, Cambridge, UK
gsa26@cam.ac.uk

[a] Polyhedral Structures Laboratory, School of Design, University of Pennsylvania, Philadelphia, USA

Abstract

The field of graphic statics focuses on the development of geometric methods to facilitate and seek optimal solutions for structural design. Such methods include the construction of reciprocal form and force diagrams that can be used as a basis for form finding tools. Building on the work of Rankine, Akbarzadeh has made significant contribution on expanding analogies between 2D and 3D extraction techniques of reciprocal diagrams to explore three-dimensional equilibrium. Additionally, McRobie has further extended Rankine's approach using geometric algebra with the notable introduction of the Corsican sum. Following the theme of the symposium, "Creativity in structural design", we explore a method for novel form finding through 3D graphic statics with the innovative use of the Corsican sum. We construct a hypothetical system of parallel forces in 3D with given boundary conditions on a plane and we intend to explore form and its corresponding force diagram. The final 3D form derives from member directions resulting from the Corsican sum in such a way that global equilibrium is accomplished given that the focus is on axial only forces. We begin by investigating a triangulation method that distributes the applied forces to the supports and gives the 2D projection of the form. We continue with the shifting of the form pieces in 2D in order to construct the vertical force diagram. Boundary conditions and equilibrium requirements guide the construction of the final 3D reciprocal force diagram. This leads to angle values for each of the structure's members. We compare the final axial forces with those resulting from alternative triangulations to proof the validity of our pattern logic.

Keywords: graphic statics, 3D equilibrium, form finding, structural design, structural optimization, Rankine reciprocal diagrams, Corsican sum, triangulation patterns.

1. Introduction

1.1 Graphic Statics

The field of graphic statics focuses on the deployment of geometric methods to visualize forces on structures. The foundations of these techniques were laid by Leonardo da Vinci and Galileo Galilei and were further developed by Varignon [1], Culmann [2], Cremona [3], Bow [4], Müller-Breslau [5], as well as by Rankine [6] [7] and Maxwell [8] [9] among others. The intuitive character of geometry constructions that guide the design of optimal structures through the reciprocity between form and force diagrams is of fundamental importance to the graphic statics doctrine. The development of computer-aided design tools since the late 1990's has led to a significant come back of graphic statics methods for training engineers and architects as well as for optimization purposes at early design stages [10]. Notable publications since the re-emergence of the field include Allen and Zalewski [11] [12], Block [13], as well as block and Ocshendorf [14]. The duality between form and force has been a useful tool for form finding strategies that have led to the development of a new architectural repertoire.

1.2 Scope

1.2.1 Objectives

This paper suggests a form finding method that follows key graphic statics techniques. Those include, but are not limited to, Akbarzadeh's [15] [16] subdivision of force diagrams, and McRobie's Corsican sum [17] [18]. Our method is based on the observation that although the Corsican sum may be used for analysis purposes, its prior force diagram construction can be used to introduce 3D form information itself. Since the Corsican sum construction is a two-step process starting with a given 2D topology, our very first step is to pursue a 2D form by a different technique. This is done through a method of subdivision of the 2D force diagram that results in an optimal 2D form. Therefore, our objectives include the construction of an optimal 2D form and the proof of the hypothesis that the force diagram of the Corsican sum can be used to extract the former pattern's 3D form diagram. We choose a simple example to explore our method that consists of 3 support points and 2 load axes.

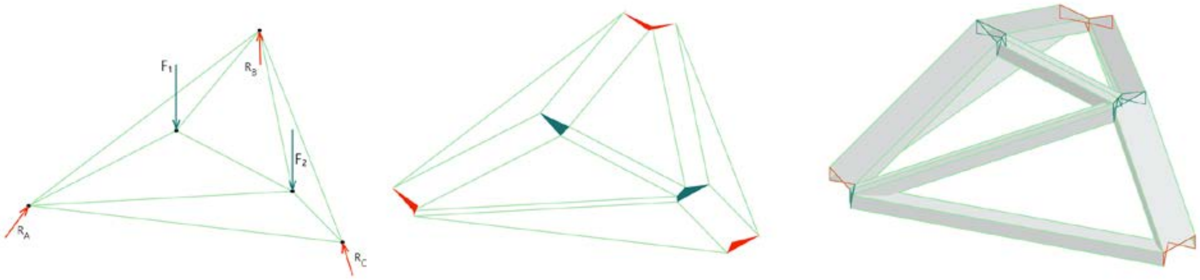

Figure1.2.1: Critical snapshots of paradigm's steps: a) initial typology hypothesis, b) Corsican sum of vertical force equilibrium, and c) Corsican sum in 3D - boundary force diagrams dictate member axes.

1.2.2 Methodology

Figure 1.3.2 illustrates the basic steps of our method and their end-to-start relationships. We begin by altering a two load axes problem to the forces' resultant solution. The resultant's force polygon is subdivided to force polygons that have equal areas to the applied loads. The number of edges of each polygon of force suggests the number of connecting bar members to its corresponding node. Force polygons are then redrawn as reciprocal (perpendicular) to their connecting members. Form pieces based on the hypothesis and force diagrams are combined to express vertical equilibrium. We then introduce horizontal force at the boundaries. This allows for the complete construction of force diagrams at the nodes based on the Corsican sum. These consist of gauche force polyhedra that provide the directions of axial forces. Assuming we are seeking axial-only form solutions, resulting directions align with the orientation of the structure's members. Finally, the process is repeated for an alternative scenario of a different initial 2D form. Resulting axial forces are then compared to validate the 2D form finding technique.

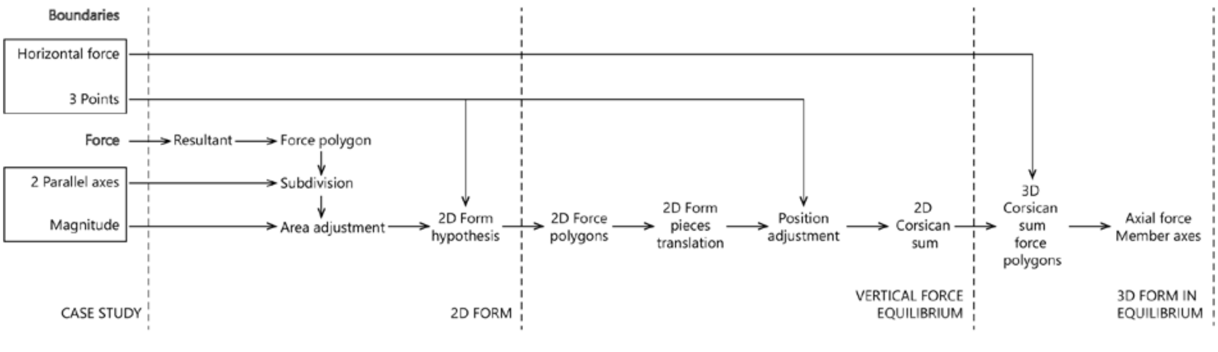

Figure 1.2.2: Method flowchart demonstrating end-to-start relationships of the introduced steps.

2. Background

2.1 Rankine reciprocal force diagrams and subdivision methods

In the study of self-stresses for pin trusses and following Rankine's principle of equilibrium on polyhedral frames [6] [7], Rankine and Maxwell [8] [9] developed graphic statics techniques that treated force diagrams as projected faces of three-dimensional polyhedra. Three dimensional implications of Rankine's and Maxwell's fundamental contribution to graphic statics have been recently expanded by Akbarzadeh et al. [19] [20]. Additionally, in translating form into a result of force equilibrium, Akbarzadeh has expandingly pursued the subdivision of the force diagram to quest form potential. Figure 2.1 demonstrates simple 2D exercises of such explorations as shown on Akbarzadeh et al. [15].

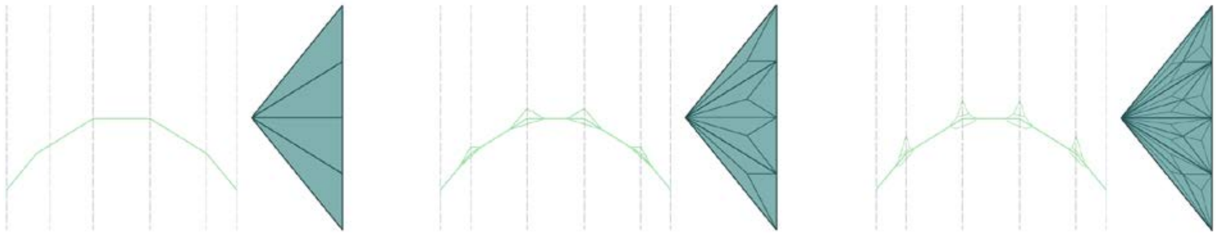

Figure 2.1: A 2D example of force polygon subdivisions resulting to several form geometries, as shown on Akbarzadeh et al. [15].

2.2 The Corsican sum

Building upon Maxwell, Rankine and others, McRobie introduced graphic statics methods that are based on the use of geometric algebra on n-dual form polytopes [17] [18]. Translation of form polygons for 2D force diagrams construction was firstly introduced in McRobie [21] with Maxwell-Minkowski sums. Splitting a given form diagram into pieces and shifting them in multiple dimensions by following a set of rules results into the construction of force diagrams in three dimensions. Notably, a similar approach of translating form diagram pieces to create a so-called 'unified diagram' has also been introduced by Zanni and Pennock [22]. Figure 2.2 demonstrates a simple example of the Corsican sum methodology for a four-panel roof as shown on Athanasopoulos and McRobie [23].

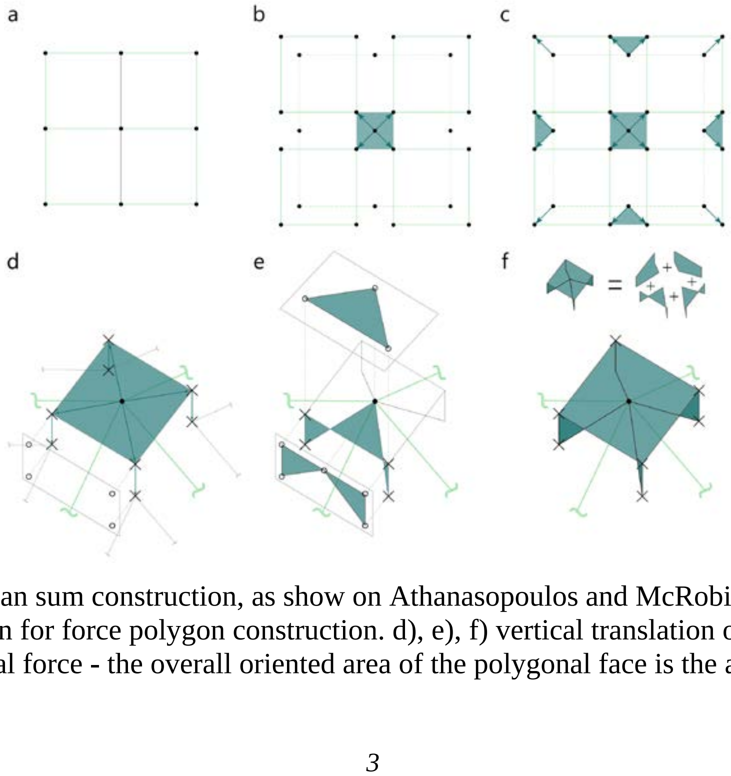

Figure 2.2: Corsican sum construction, as show on Athanasopoulos and McRobie [22]: a), b), c) 2D form translation for force polygon construction. d), e), f) vertical translation of form pieces to introduce horizontal force - the overall oriented area of the polygonal face is the axial force direction.

3. Application

3.1 Constructing 2D form projection

3.1.1 Example setting and force resultant

Our case study structure consists of three points ABC and two parallel axes of applied loads perpendicular to the plane defined by the boundaries. We position F_1 and F_2 loads of any magnitude anywhere along the two axes. At this original formation, there are no member bars connecting the boundary vertices to internal points, there is only an outer triangle. The objective is to seek an optimal configuration of structural members that will carry axial-only forces from the internal parallel axes to the external corners. Wolfe's [24] methods are used to determine the resultant for parallel forces graphically. We project F_1 on F_2 axis and vice versa. Connecting the projections endpoints in reverse creates a gauche polygon whose self-intersecting point O lies on the resultant's axis. The magnitude is then the sum of the two applied forces. Projecting point O on the ABC plane results in a form diagram for F_T that is used to extract the reciprocal force diagram by using Corsican sum, shown on figure 3.1.1. This is drawn by shifting the form pieces on ABC plane along vectors starting from O´ and have a perpendicular direction to AB, BC, and AC bars. The area of the polygon created by the moving vectors equals F_T. Additionally, each force triangle area equals to the vertical force to be carried by its corresponding bar.

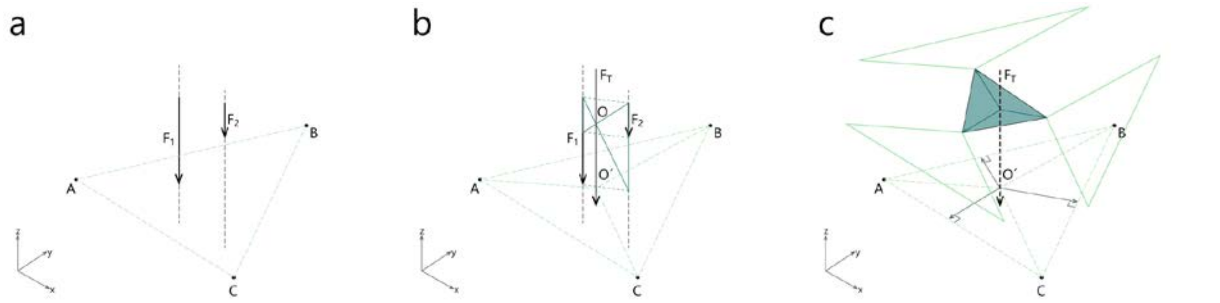

Figure 3.1.1: a) example setup configuration – three boundary points with two load axes, b) determining parallel forces resultant following Wolfe's crossing and intersecting method – projecting each force to the axis of the other one and then connecting their endpoints in reverse, c) construction of force polygon based on exterior product of moving vectors, each vector is perpendicular to the outer edge of its corresponding triangle – final diagram elevated on the z-axis for clarity.

3.1.2 Resultant's force polygon subdivision and form hypothesis

Applied load axes intersect ABC plane at points pt_1 and pt_2. Connecting these points draws the projection of the mid bar member of the final structure the angle of which remains an objective of our method. Since the Maxwell convention is followed, the axial force of the mid bar is drawn on the axis perpendicular to it. This axis is now used to split the resultant's force polygon into two polygons each one corresponding to a node. Since any construction method of the reciprocal force diagram does not necessarily predict its final position, resulting split polygons have arbitrary areas. We choose to position the axis in a way that polygon areas equalize the forces applied to their corresponding nodes (fig. 3.1.2). The desired position of the axis is trivially found by using Thales' similarity theorem. This is applied initially on the three-sided polygon which is scaled following Thales to match the desired area. Each polygon's count of edges suggests the number of connecting members, or connected vertices, to its corresponding node in 2D. The directions of these members, or the vertices to connect to, are based on proximity of the nodes to the extended perpendicular bisectors of the polygon edges. The resulting 2D form configuration is shown on figure 3.1.2. Additionally, force polygons need to be redrawn so that their edges are perpendicular to their reciprocal bars.

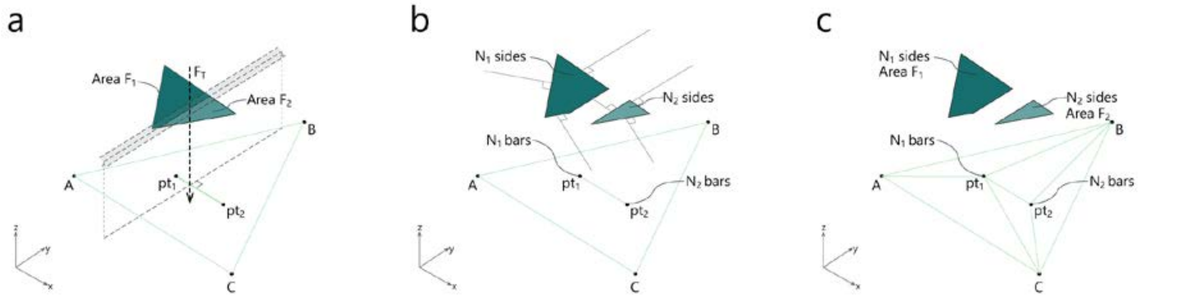

Figure 3.1.2: a) resultant's force polygon splits to create two force polygons each that have areas equal to applied forces – position adjustment can be done by following Thales' similarity theorem starting with the triangular polygon b) the number of edges of each polygon recommend the number of bars to be connected to their corresponding node c) the final drawing of the form structure to proceed with.

3.1.3 Corsican 2D

Planar form diagram is established, and we proceed to seek its 2D Corsican sum force diagram to visualize vertical force equilibrium (fig. 3.1.3). Beginning with points pt_1 and pt_2 we intend to construct force polygons with areas equal to their applied loads. Polygon edges must be perpendicular to the member bars connecting at each node. Following the Corsican sum doctrine, we choose to construct the polygons by using vectors to later shift the form pieces. A similar process is followed here as in Athanasopoulos and McRobie [23]. Clearly, for this configuration, and depending both on the scale of force and the vectors starting point, self-intersecting polygons may emerge. However, for our limited scope we proceed with selections that eliminate this. Figure & shows the translated form pieces based on the force polygons created. The "Maxwell-Minkowski"-like figure now needs to be shifted such that all parallelograms created are rectangles. This is done by using the outer bars' perpendicular bisectors for the resultant's solution and the two loads case. The intersection point of the bisectors must be the same in both cases. This changes the starting points of the force polygon vectors, however the distance of pt_1 and pt_2 remains the same. Vectors are then moved to the boundary points and the reactions' polygon pieces are created. The sum of their areas equals the sum of applied loads. This manifests vertical force equilibrium.

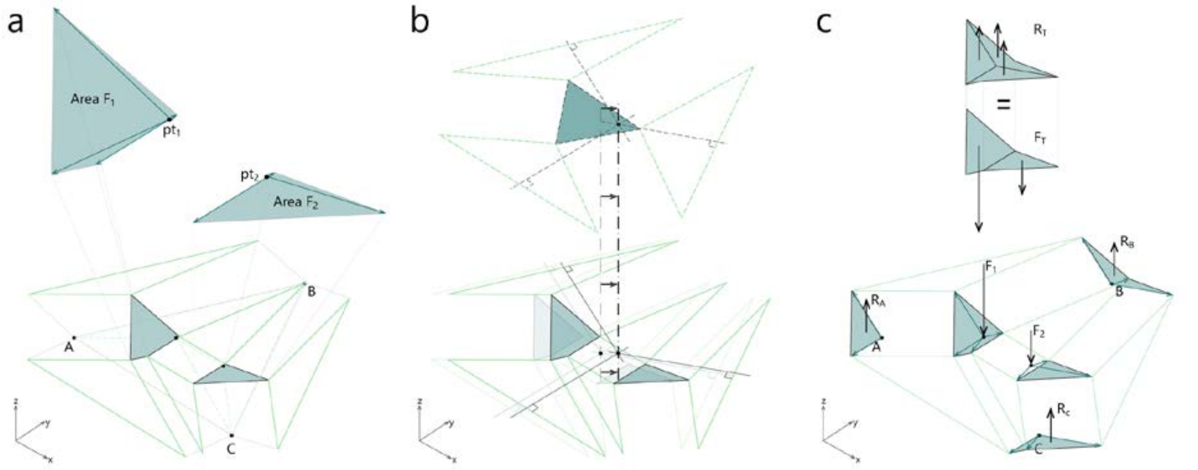

Figure 3.1.3: a) redrawing of force polygons so that their edges are perpendicular to their corresponding bars, b) repositioning form and force diagram so that all parallelograms created are rectangles, c) areas of polygons created at the boundaries equal reaction forces magnitudes – a simple overlay of reaction polygons onto applied force polygons manifests vertical equilibrium.

3.2 Introducing horizontal force

The Corsican sum diagram construction is completed with the assembly of the pieces representing horizontal force. Figure 3.2.1 demonstrates the translation of form pieces on the z axis so that moving vectors generate gauche polygons. The areas of the orthogonal projections of these geometries represent horizontal forces. Since the horizontal force of the outer ring member bars is known, drawing begins by moving their corresponding form pieces such that their bilateral perpendicular polygons represent the magnitude of their axial forces. Nodal equilibrium is expressed if the gauche faces of a node's force diagram sum up to a closed geometry. Trivially, this means that the rest of the form pieces corresponding to the inner bars are vertically shifted by the same height. This can result in nodal force polygons that indicate nodal equilibrium in all directions. Figure 3.2.1b illustrates how nodal force diagrams manifest equilibrium in all axes with the exception of the projections of the polygonal faces that correspond to the internal bars.

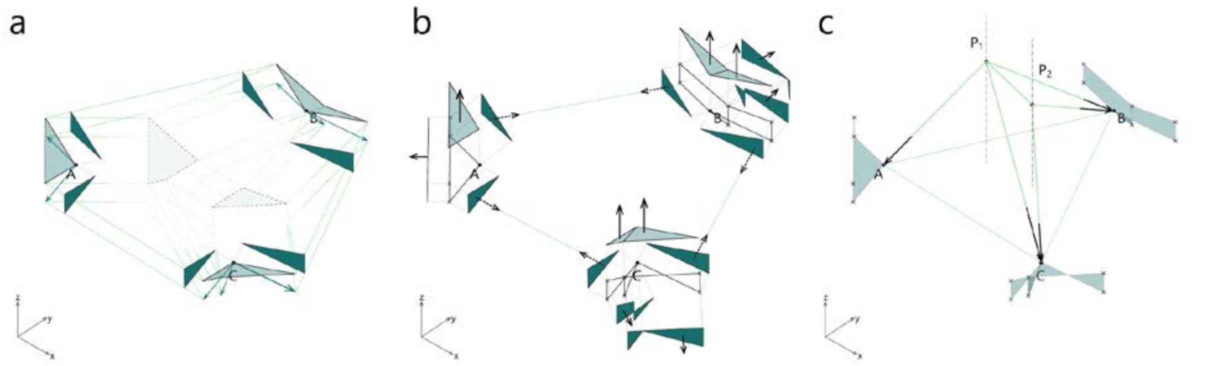

Figure 3.2.: a) z-axis translation of support location's form polygons to introduce horizontal force for the outer bars – the orthogonal projections of the gauche polygons represent horizontal forces, b) total areas of projected polygons equal to zero excluding the forces carried by the internal bars, c) polygonal faces corresponding to the inner bars have an overall oriented area that represents the axial forces directions – cross intersections of the introduced axes with the applied loads axes validate the method.

Reciprocal force diagrams of the Corsican sum are built. Each node's force diagram consists of a number of gauche faces equal to the count of members connecting to the node. The overall oriented area of each face is the direction vector of the member since we limit our approach to axial-only forces. Directions of members are easily found by adding the vectors of each faces orthogonal projections. Extending the axial forces lines of action completes the form diagram. Evidently, resulting member axes co-intersect together with the applied load axes on points P_1 and P_2 as shown on figure 3.2.1c. This demonstrates the validity of the method followed.

3.3 Alternative pattern

Figure 3.3 shows two different Corsican sum solutions. Axial forces are represented as the extrusion geometries' cross sections. Intuitively, volumetric appearances demonstrate axial stresses. The Corsican sum solution shown on figure 3.3b is based on the subdivision method for the resultant's force diagram formerly described, while figure 3.3d begins with an alternative 2D topology. In both cases, the Corsican sum force diagrams provide member directions that intersect the applied load axes. Therefore, the assumption that this doctrine's methods, dual polytopes and vector exterior products, can be used in reverse for the design of form resulting from force validate itself, for our case of non-self-intersecting force polygons and for the study of axial only forces. Furthermore, a comparison of the sums of the resulting axial forces of the members suggests that our topology exploration method, which was based on the subdivision of the force polygon of the resultant, delivers optimal solutions for our case scenario. Such a remark must be further addressed in the future.

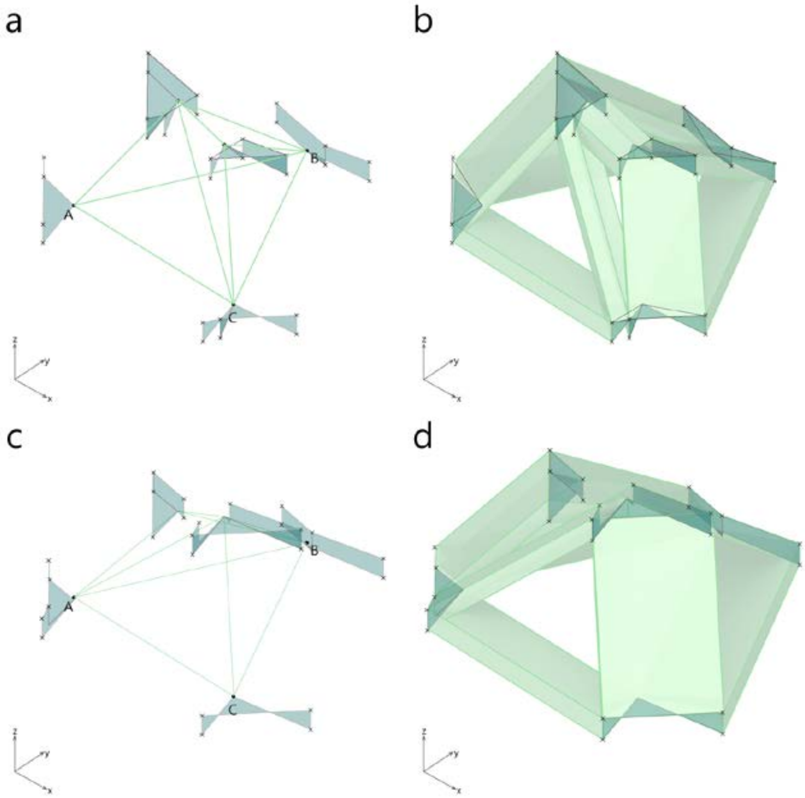

Figure 3.3: Comparison of axial forces for the two possible triangulation patterns a), b) form diagram and Corsican sum based on the methodology followed, c), d) alternative form diagram and Corsican sum – cross sections areas of volumes indicate the different axial forces.

4. Conclusions and future work

We introduced a method for determining a 2D connectivity pattern for a case of three supports and two nodal applied forces. The main principle of the pattern design followed a strategy of the subdivision of the force polygon of the force resultant. We showed that this pattern can lead to optimal design solutions for axial forces. This was done using the Corsican sum for the case of non-self-intersecting 2D force diagrams for vertical forces. Future directions of the approach may address the question on how self-intersecting force polygons may be avoided by determining alternative 2D patterns. The determination of 2D topologies where Rankine reciprocal force polygons can exist. Furthermore, future applications may include multiple parallel loads as well as multiple support locations for applying the same methodology for axial only forces. Finally, it was shown that the construction of Corsican sum's nodal polygon of forces in advance may be used to extract form information. Therefore, for Corsican sum form polygons that start with valid area planar polygons can directly be used to extract information on the axial forces of members at each node.

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