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Towards a novel form-finding approach using matrix analysis: exploiting nodal displacements of pin-jointed frameworks

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Abstract

This paper presents a simple method to calculate nodal displacements of kinematically indeterminate pin-jointed frameworks using matrix analysis, aiming to facilitate the early-stage design and analysis of space frames. This method can also inform a novel funicular form-finding approach by iteratively accumulating nodal displacements until reaching a new state of equilibrium. This method assumes that the framework is massless and inextensional, and requires only minimal inputs such as the frame geometry, support locations, and force vectors. The method intends to provide fast and preliminary engineering insights into the kinematic behavior of the framework, like identifying critical locations and suggesting new forms that best fit the force vectors. The proposed method can be a useful tool for designers in the early stages of design, providing them with quick and intuitive displacement results to inform design iterations. By using this method, designers can bridge the gap between early-stage designs and in-depth engineering rationalizations. The results can be used to identify potential problems or limitations in the design and suggest improvements or modifications to optimize the design. This method has been implemented as part of a Grasshopper® plug-in that is freely available for designers to explore. Several case studies are also demonstrated to show how this method can facilitate early-stage space frame designs.

Keywords: funicular form-finding, matrix analysis, kinetic indeterminacy, multi-layer space frame

1. Introduction

Space frames are commonly used in the engineering world at various scales due to their lightweight and high efficiency. Usually, the initial form-finding and schematic design requires the participation of experienced engineers with structural intuition and knowledge of structural analysis and material properties. The geometry-based structural design method based on 2D and 3D reciprocal diagrams developed since the 19th century [1–12], namely 2D/3D graphic statics (GS), allows designers without limited knowledge in the engineering field to create, manipulate, and analyze structural forms while knowing the internal force distributions. However, those preliminary structural forms are usually kinematically unstable and are generated solely based on the design loads, which only represent the equilibrium for this specific load case. That is, when the applied loads are different from the design loads, the structural form may have deflections. To ensure their safety and reliability in real-world scenarios with more complex boundary conditions and material properties, it is crucial for professionals to carefully examine and evaluate them using more rigorous engineering methods such as finite element analysis (FEA) before moving into design development. The design process often requires multiple iterations, and designers usually need to take the evaluation and analysis results to inform and adjust the forms.

There is a gap between the geometric-based form-finding process and the more sophisticated engineering rationalization. To bridge this gap, this paper proposes matrix analysis methods as an intermediate stage, which can provide designers with engineering insights while maintaining clarity and simplicity. During the matrix analysis, a structural form is simplified and abstracted as an easy-to-analyze pin-jointed inextensional framework, and its equilibrium matrix and kinematic matrix (also called compatibility matrix) can be constructed. Although space frames are often made with rigid joints, the behavior of a pin-jointed frame with the same configuration could be a good indicator and estimate of the performance of real engineering structures. By using matrix analysis methods, designers can obtain a wealth of static and kinematic information that cannot be abstracted from the conventional formulation without considerable effort [13]. It allows a better understanding of the structure's behavior under different load cases and empowers designers to make informed decisions throughout the design process.

Focused on the kinematic aspect of matrix analysis, this paper proposes a way to calculate the nodal displacements using minimal inputs, which can then be exploited to inform a novel approach for form-finding, aiming to facilitate the early-stage design of space frames. Nodal displacements serve as an intuitive indicator and can be used for spotting critical locations for a specific load case. Moreover, the deformed configuration often represents a more appropriate form for this load case, suggesting an approach for form-finding and modification.

1.1. Related work

1.1.1. Matrix analysis on the static and kinematic indeterminacies of pin-jointed frameworks

Matrix analysis is found convenient in organizing calculations in engineering fields such as aerospace, civil, and mechanical engineering. With a focus on pin-jointed frameworks, Calladine [14], Pellegrino and Calladine [15] formulated an algorithm that evaluates the static and kinematic indeterminacies in a rapid manner by determining the rank of equilibrium and kinematic matrices and the bases of its four linear-algebraic vector subspaces. Specifically, it offers complete details of any states of self-stress and modes of inextensional deformation that a framework may possess. Lu et al. [16] adapted this method from bars to faces and applied it to the design of multi-layer sheet-based funicular structures by ensuring all mechanisms are restrained. Pellegrino [17] later presented a matrix algorithm that analyzes the mechanisms and nodal displacements of a pre-stressed network upon change of loads, which leads to a new deformed configuration and can be used for form-finding. This algorithm requires the input of material behaviors such as Young's modulus and cross-sectional areas of each bar.

1.1.2. Methods for calculating nodal displacements and the corresponding form-finding approaches

Existing form-finding methods of space frames based on nodal displacements can be categorized into three main clusters [18], including geometric matrix methods [19, 20], stiffness matrix methods [21], and dynamic relaxation methods [22, 23]. Geometric stiffness methods are material-independent, with only a geometric stiffness that relates the forces to the length of the edge members. Stiffness matrix methods are based on the standard elastic and geometric stiffness matrices. Both geometric and stiffness matrix methods find the equilibrium by reducing unbalanced forces. Dynamic relaxation methods, on the other hand, calculate the velocity and acceleration for each time increment, until the structure becomes static due to artificial damping. A variety of inputs need to be prepared in order to use such methods. For stiffness matrix methods, the elastic properties need to be provided in order to set up the tangent stiffness matrix. For the geometric matrix methods, a prescribed set of force densities together with the applied forces need to be provided in order to find a form. The dynamic relaxation methods also require elastic properties and masses to calculate the physical movements.

1.2. Problem statement and objectives

As stated above, an intermediate stage is desired to bridge the gap between geometric-based form-finding and engineering rationalization. Analysis of the equilibrium matrix and the kinematic matrix is a great candidate as it can provide a wealth of static and kinematic information without much effort, maintaining simplicity and clarity for early-stage design iterations. However, for more detailed information such as nodal displacements, it still needs to assign material-related parameters to the framework in order to carry out calculations. This paper aims to provide a simple method for calculating nodal displacements that requires minimal input such as the structural form and force vectors on the nodes, hoping to assist preliminary space frame designs and enrich the available toolset. In other words, this method is able to qualitatively analyze the nodal displacements given a user-defined framework with support conditions. The results can be used in the following ways:

- Display the tendency of nodal displacements if the applied force vectors cannot be balanced.
- Identify critical locations with large displacements.
- Accumulate the displacements incrementally, serving as an accurate kinematic simulation tool.
- Iteratively reverse and accumulate the displacements towards a different state of equilibrium, suggesting a new form that better fits this load case.

2. Method

This section explains in detail how the nodal displacements are calculated using matrix analysis with the inputs of a 3D framework, a group of force vectors associated with the nodes, and support locations. It requires three simple steps: constructing the equilibrium matrix and kinematic matrix for the given framework and support locations; calculating unbalanced forces given the force vectors; calculating nodal displacements based on the unbalanced forces.

2.1. Construction of the equilibrium matrix and kinematic matrix

The construction of the equilibrium matrix and kinematic matrix follows the convention established by Pellegrino and Calladine [15]. The framework is described as b bars (or edges) pin-jointed by j nodes (or vertices) with k supported dimensions, each supported dimension defined as constrained movement along one direction. Two kinematic variables and two static variables are considered, they are displacements of the vertices assembled in a $[3j-k \times 1]$ vector $\mathbf{d}$; elongation coefficients of the edges assembled in a $[b \times 1]$ vector $\mathbf{e}$, each entry defined as $e_i \times l_i$ where e_i is the elongation of the i-th edge, l_i is the edge length; external forces on the vertices assembled in a $[3j-k \times 1]$ vector $\mathbf{f}$; and tension densities in the edges assembled in the $[b \times 1]$ vector $\mathbf{t}$, each entry defined as t_i/l_i where t_i is the tension of the i-th edge. The elongation coefficients and tension densities can be easily turned to the actual elongations and tensions as the edge lengths are known. All forces applied to each node must be in equilibrium, written as

$$\mathbf{A} \cdot \mathbf{t} = \mathbf{f}, \tag{1}$$

where $\mathbf{A}$ is the $[3j-k \times b]$ equilibrium matrix. And all extensions in the edges must be compatible with the nodal displacements, written as

$$\mathbf{B} \cdot \mathbf{d} = \mathbf{e}, \tag{2}$$

where $\mathbf{B}$ is the $[b \times 3j-k]$ kinematic matrix, or compatibility matrix. It's easy to prove by the principle of virtual work that the equilibrium matrix and kinematic matrix are the transposes of each other [14]

$$\mathbf{B} = \mathbf{A}^{\mathbf{T}}. \tag{3}$$

2.2. Calculation of unbalanced forces

Eq.1 is helpful in terms of calculating the forces in the edges given $\mathbf{f}$. However, not all $\mathbf{f} \in \mathbb{R}^{3j-k}$ can be equilibrated by the framework. According to Pellegrino and Calladine [15], the framework will only be in equilibrium when **f** lies in the column space of the equilibrium matrix **A**. Otherwise, mechanisms (pin-joints) will be activated when it lies in the left null space of **A**. In such cases, assuming that the framework has no mass, then the least squares solution of **t**, denoted as $\hat{\mathbf{t}}$, can be used to estimate the edge forces, calculated as

$$\hat{\mathbf{t}} = (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{f} \tag{4}$$

when **A** has a full rank. If not, $\hat{\mathbf{t}}$ can be calculated as

$$\hat{\mathbf{t}} = \mathbf{A}^{+}\mathbf{f}, \tag{5}$$

where $\mathbf{A}^{+}$ denotes the Moore-Penrose Inverse of **A**.

The principle of least action can provide some evidence for it. The action S of the framework is defined as the integral of the Lagrangian L between time t_1 and time t_2, written as

$$S = \int_{t_1}^{t_2} L(t)dt \tag{6}$$

where L is the difference of the kinetic energy T and potential energy V.

$$L = T - V \tag{7}$$

For the framework, it's assumed as particles connected by massless bars. Let the mass of each particle be m, then T and V can be written as

$$T = \frac{1}{2}\sum_{a=1}^{3j-k} mv_a^2 \tag{8}$$

$$V = \sum_{i=1}^{j-k_z} mgz_i \tag{9}$$

where v_a is the velocity of the particles along each 3D Euclidean axis, z_i denotes the z-coordinate the the i-th particle, and k_z denotes the number of nodes that have their z-coordinate constrained.

Since the whole system is not in equilibrium, the velocity v_a will be determined by the summation of all external forces and all axial forces in the edge members, namely residual forces or unbalanced forces and assembled in a $3j$ - k vector $\mathbf{f_r}$, written as

$$\mathbf{f_r} = \mathbf{f} - \mathbf{A}\hat{\mathbf{t}} \tag{10}$$

where external forces **f** and equilibrium matrix **B** are known. Then the Lagrangian of the middle point is used to approximate the Lagrangian for the small time interval dt. The vertex velocities, assembled in a $3j$ - k vector **v** can be written as

$$\mathbf{v} = \frac{\mathbf{f_r}}{2m}dt. \tag{11}$$

Then, we can solve for the vertex velocity **v** of the framework by minimizing the Lagrangian of the small time interval

$$\underset{\mathbf{t}\in\mathbb{R}^e}{\text{minimize}}\ \mathbf{L} = \mathbf{T} - \mathbf{V} = \frac{1}{2}\sum_{a=1}^{3j-k} mv_a^2 - \sum_{i=1}^{j-k_z} mgz_i = \frac{dt^2}{8m}\mathbf{f_r^2} - \sum_{i=1}^{j-k_z} mgz_i. \tag{12}$$

When the mass of the vertex particles are infinitely small, then the kinetic energy in the Lagrangian becomes dominant

$$\underset{\mathbf{t}\in\mathbb{R}^e}{\text{minimize}}\ \mathbf{L} = \frac{dt^2}{8m}\mathbf{f_r^2}. \tag{13}$$

which is equivalent to solving for the least square solution $\hat{\mathbf{t}}$ of Eq. 1, proving that the residual forces $\mathbf{f_r}$ can be calculated by Eq. 10. When the frame is kinematically determinate, the kinematic indeterminacy m equals zero

$$m = 3j - k - r_{\mathbf{A}} = 0, \tag{14}$$

where the rank of $\mathbf{A}$ is denoted as $r_{\mathbf{A}}$. This shows $\mathbf{A}$ has a full rank and Eq.1 has a solution. In this case, the unbalanced forces will be zero.

2.3. Transforming unbalanced forces to nodal displacements

As mentioned by Pellegrino [17], any infinitesimal inextensional displacement from the original configuration, while leaving the prestress unchanged, results in unbalanced loads of magnitude proportional to the size of the displacement. This is also true in our case when restress does not exist. That means the nodal displacements in the inextensional mode can be directly represented as the residual forces $\mathbf{f_r}$ multiplied by a user-specified scale factor ξ

$$\mathbf{d} = \xi\mathbf{f_r}. \tag{15}$$

This can be proved using Eq. 2

$$\mathbf{Bd} = \xi\mathbf{Bf_r}. \tag{16}$$

Substitute $\mathbf{f_r}$ using Eq.10,

$$\mathbf{Bd} = \xi\mathbf{B}(\mathbf{f} - \mathbf{A}\hat{\mathbf{t}}). \tag{17}$$

Then substitute $\hat{\mathbf{t}}$ using Eq.4, and substitute $\mathbf{B}$ using Eq.3,

$$\mathbf{Bd} = \xi\mathbf{A^T}(\mathbf{f} - \mathbf{A}(\mathbf{A^TA})^{-1}\mathbf{A^Tf}) = \xi(\mathbf{A^Tf} - \mathbf{A^TA}(\mathbf{A^TA})^{-1}\mathbf{A^Tf}) = \xi(\mathbf{A^Tf} - \mathbf{A^Tf}) = \mathbf{0}, \tag{18}$$

which shows zero edge elongation coefficients, meaning that the nodal displacements are under inextensional mode. That is, there won't be any edge elongation during the displacement, and the result can be a good estimation of the behavior of pin-jointed rigid bars. Note that the above calculations are based on the small displacement theory. The displacement results are accurate when $\xi\mathbf{f_r}$ is small. Otherwise, matrices $\mathbf{A}$ and $\mathbf{B}$ are no longer valid and need to be updated.

2.4. Accumulation of the small displacements

Given the applied force vectors are unchanged and always attached to the nodes, the small displacements can be incrementally accumulated until the frame reaches a new form where all applied force vectors can be balanced and there is no force residual. This process simulates the more accurate kinematic behavior of the frame in response to the applied forces. Note that matrices $\mathbf{A}$ and $\mathbf{B}$ need to be updated after each step. The updated equilibrium matrix and kinematic matrix are denoted as $\mathbf{A}'$ and $\mathbf{B}'$, respectively. Again, when $\xi\mathbf{f_r}$ is small the displacements are accurate at each step and the change of edge lengths is negligible. The Moore-Penrose Inverse can be used again to determine whether the applied forces are all balanced. The deformed frame reaches a stable state only when Eq.19 holds:

$$\mathbf{f} = \mathbf{A}'\mathbf{A}'^{+}\mathbf{f}. \tag{19}$$

2.5. Accumulation of the reversed displacement: towards a novel form-finding approach

Instead of deforming along the displacement direction, the frame can also deform opposite to the displacement direction at each step, acting like a pre-deformation to counteract the displacements. This iterative operation is the same as the process described in Section 2.4, except that the displacements applied to the frame are reversed at each step. After reaching a new state of equilibrium, the new frame geometry will represent a new form that fits the applied load. The computational pipeline of those processes are illustrated in Figure 1.

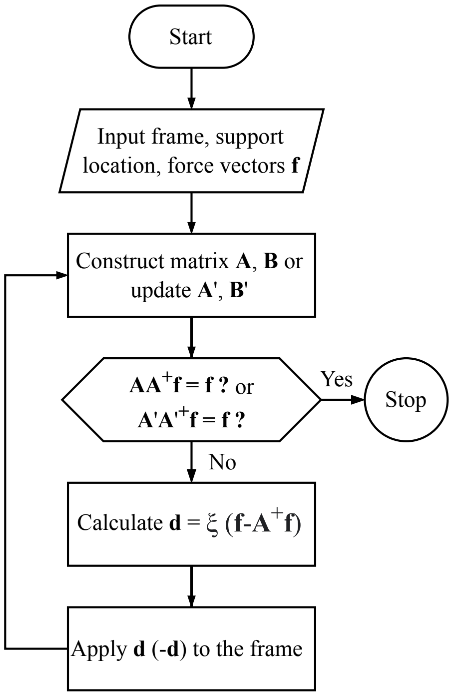

Figure 1. The computational flowchart of incremental accumulation of the (reversed) displacements.

3. Case study

The above-mentioned process has been implemented and included as part of a Grasshopper® plug-in named PolyFrame 2 [24]. The matrix computations are supported by Numpy.NET [25], a .NET binding of the scientific computing library Numpy [26]. With the help of this implementation, several studies are created to demonstrate the possible use cases.

3.1. Deformation estimation

Although this proposed method for calculating displacements is based on many assumptions such as massless frame, frictionless pin-joint, and rigid edge, the result can still represent a qualitative estimation of the real behavior of a structure. Figure 2 shows the comparison of analysis results between this proposed method and the finite element method (FEM). For the FEM, three different materials are simulated. The results manifest clear relevance between the simple nodal displacement results and the more rigorous FEM results.

3.2. Simulate the kinematic behavior or find a new form

Figure 3 shows the incremental accumulation of the nodal displacements from an initial flat grid network. In each step, if the displacements are applied to the previous state of the frame, the result will be a

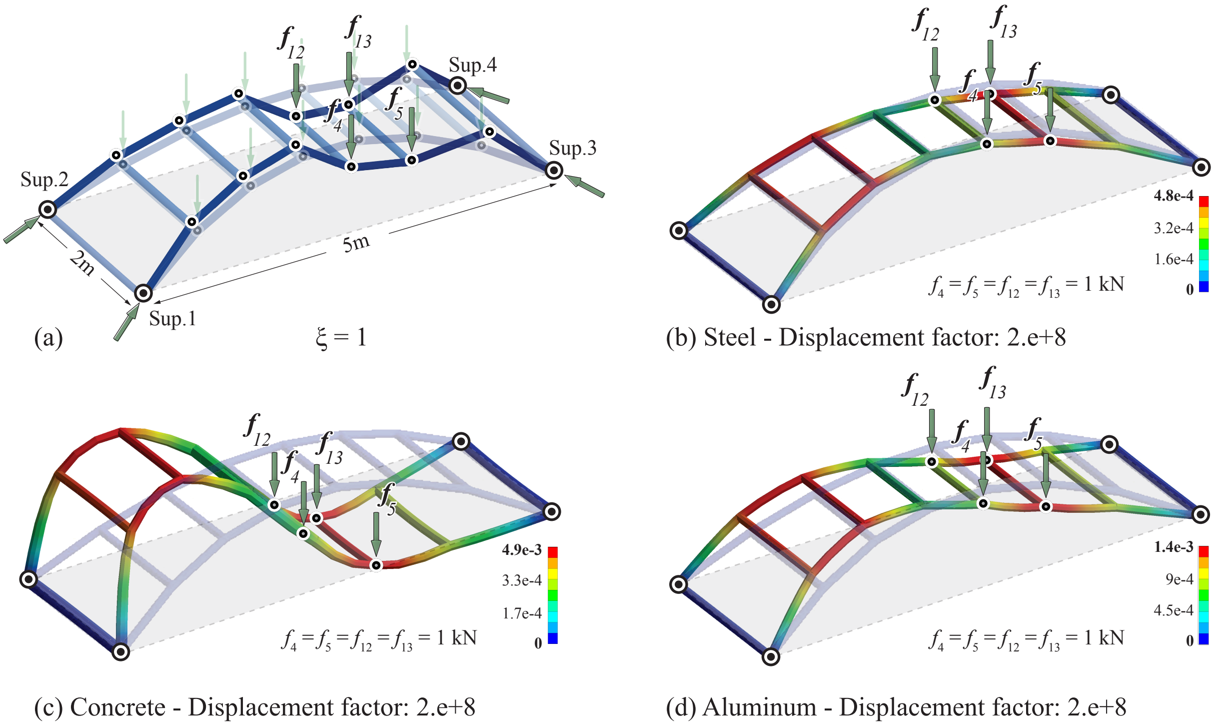

Figure 2. Compare the nodal displacement result with FEA.

kinematic simulation in response to the external forces (Figure 3a); otherwise, the result will show a new form that better fits the applied forces (Figure 3b, 4).

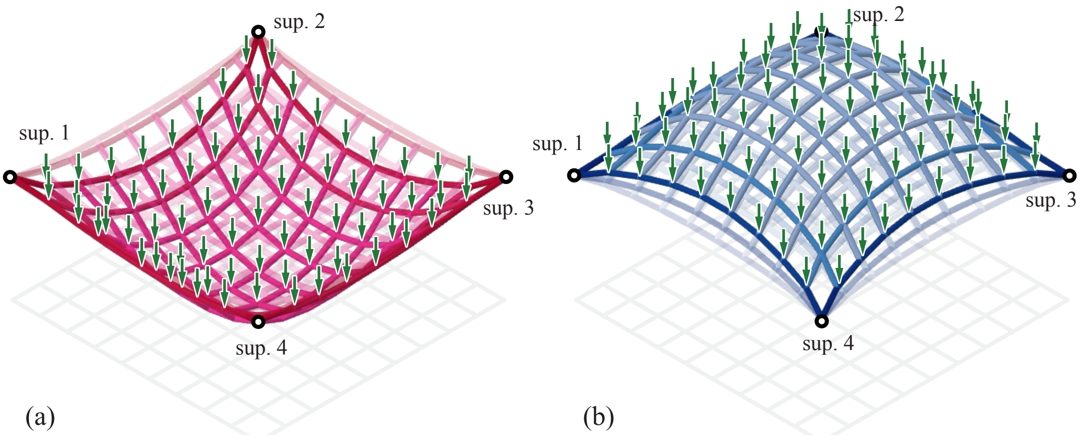

Figure 3. (a) The nodal displacements are iteratively applied to the frame, (b) The reverse of the nodal displacements are iteratively applied to the frame.

4. Conclusion

This paper presents a simple method to calculate nodal displacements of a pin-jointed framework under a group of force vectors using matrix analysis. This method assumes that the framework is massless and inextensional, and requires only minimal inputs such as the frame geometry, support locations, and force vectors. The method intends to provide fast and preliminary engineering insights into the kinematic behavior of the framework, like identifying critical locations and suggesting new forms that best fit the force vectors. The proposed method can be a useful tool for designers in the early stages of design, providing them with quick and intuitive displacement results to inform design iterations. By using this method,

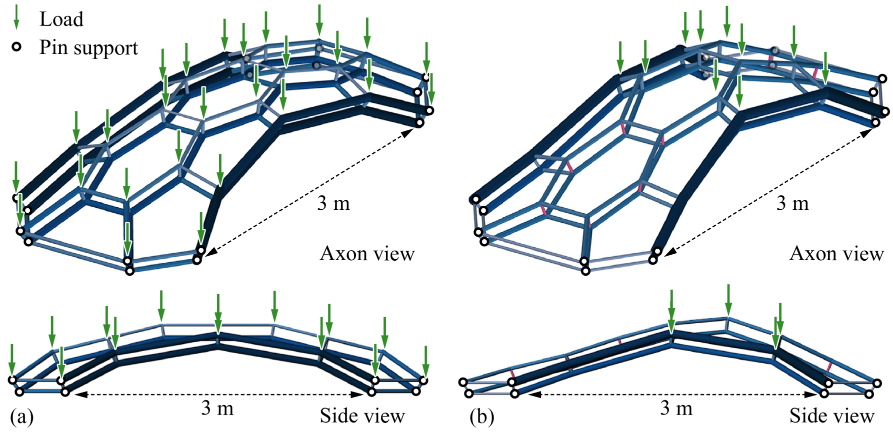

Figure 4. By applying the reverse of the asymmetrical forces, a new bridge geometry can be found that better fits the new load case.

designers can bridge the gap between early-stage designs and in-depth engineering rationalizations. The results can be used to identify potential problems or limitations in the design and suggest improvements or modifications to optimize the design. This method has been implemented as a software tool that is freely available for designers to explore. Several case studies are also demonstrated to show how this method can facilitate early-stage space frame designs.

In summary, the proposed method enriches the toolset for designing space frames, allowing designers to make informed decisions based on preliminary engineering analysis, and saving time and resources in the design iteration process.

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