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3D auxetic materials designed with algebraic polyhedral graphic statics

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Abstract:
Auxetic materials show unusual reaction behaviors under uni-axial compression or tension forces. For instance, they contract perpendicular to the load direction under an applied compressive force and expand under tension. Their unique behavior has been harnessed in the fields of sensors, medical instruments, sportswear, aerospace devices, etc. As a continuation of the previous research that proposes to use algebraic 2D graphic static for the design of 2D auxetic metamaterials, this paper presents a novel approach to designing 3D auxetic metamaterials using an algebraic formulation of polyhedral reciprocal diagrams (PGS), resulting in concave geometric configurations that are shared by many auxetic materials. The proposed method offers two approaches for generating these concave configurations: modifying a convex polyhedral framework using geometric transformations or constructing a concave polyhedral framework directly using the algebraic formulation of PGS. The validity of the proposed method is demonstrated by testing two models using the finite element method, which clearly shows a negative Poisson's ratio.

Keywords: auxetic materials, negative Poisson's ratio, three-dimensional graphic statics, algebraic graphic statics, reciprocal polyhedral diagrams

1. Introduction

Auxetic metamaterials show unusual reaction behaviors under uni-axial compression or tension forces. They contract perpendicular to the load direction under an applied compressive force and expand under tension. This unique behavior, namely negative Poisson's ratio, has found its way to be applied to various fields such as sensors, medical instruments, sportswear, aerospace devices, etc [1]. Reid et al. [2][3] stated that this behavior heavily depends on the internal concave cellular geometries. Existing methods for designing auxetic materials are mostly empirical, meaning that people usually sculpt the configuration based on experiences as well as intuitions.

A previous research project aims to provide a more systematic approach to designing 2D auxetic materials using algebraic 2D graphic statics (A2DGS) [4]. Focusing on the geometric formulation, the algebraic method has shown its ability to create a variety of 2D auxetic patterns. This paper intends to extend this design method to three dimensions through the algebraic formulation of polyhedron-based 3D graphic statics (3DGS), also called polyhedral graphic statics (PGS) [5], hoping to provide a tool and help the science, design, and engineering communities investigate and develop new auxetic metamaterials with ease.

1.1. Related work

1.1.1. Current design approaches of auxetic materials

Auxetic behaviors exist in both natural and man-made materials. Within the scope of man-made materials, there are various types of geometrical configurations, including re-entrant models [6–10], rotating polygonal models [11–14], chiral models [15–19], crumpled sheets models [20, 21], perforated sheets models [22–24], random network models [2, 3], etc. Among those different geometrical configurations, some are periodic cellular, and others are irregular and disordered. The discoveries and reports of new auxetic models usually lack a general guideline. Reid et al. [2][3] observed that the concave geometric configuration is a common characteristic of the auxetic metamaterials, upon which a computational algorithm was developed to generate random network-based patterns with negative Poisson's ratio.

1.1.2. Algebraic formulation of graphic statics and its application in auxetic materials

Graphic statics is a geometric form-finding tool based on reciprocal form and force diagrams. Hablicsek and Akbarzadeh [4] found the algebraic formulation of 2DGS convenient in tuning the global configuration of form diagrams based on the geometric degrees of freedom (GDoF), which can be used to create 2D auxetic patterns. The polyhedral nature of PGS makes it a great tool for creating cellular geometries. The algebraic formulation for PGS has also been formulated by Hablicsek et al. [25], Akbarzadeh and Hablicsek [26, 27], and Lu et al. [28] proposed an improved formulation with comprehensive edge length and vertex location controls for the polyhedral reciprocal diagrams, providing a handy tool for generating 3D cellular geometries with concave configurations.

1.2. Problem statement and objectives

As stated above, the previous explorations and discoveries of auxetic metamaterials are case specific. Some explorations have provided a systematic approach in 2D. By extending the approach developed for 2D materials to 3D, this paper aims to propose a new design method that can create more complex and sophisticated geometries with concave configurations that exhibit negative Poisson's ratio. One of the major challenges in this approach is ensuring that the resulting configurations are truly concave without any entangling and self-intersecting, which is addressed by exploiting the power of the enhanced algebraic formulation of PGS.

2. Method

In the context of PGS, a form diagram is usually generated given an input force diagram. There are two existing implementations of PGS, an iterative approach based on nodal operations [29], and an algebraic approach based on the mathematical essence of the form and force reciprocity [25–28]. The iterative approach is limited solution space due to the geometrical restrictions introduced by the iterative algorithm, which makes it difficult to create concave cellular geometries. Focused on the more abstract mathematical relationships, the algebraic method has a greatly extended solution space that enables the flexible exploration of the GDoF, making it easier to generate geometries with concave cells. This section is based on the enhanced algebraic formulation with comprehensive edge length and vertex position control that has been extensively described by Lu et al. [28]. Since only the edge length control is used in this paper, this formulation can be simplified as a system of non-homogeneous constrained linear equilibrium equations written as follows:

$$\mathbf{M}\mathbf{q} = \mathbf{t}. \tag{1}$$

Here $\mathbf{q}$ is a vector that contains all edge lengths in the form diagram, and the matrix $\mathbf{M}$ is named the constrained equilibrium matrix, obtained by vertically stacking the equilibrium matrix $\mathbf{A}$ and the edge constraint matrix **B**:

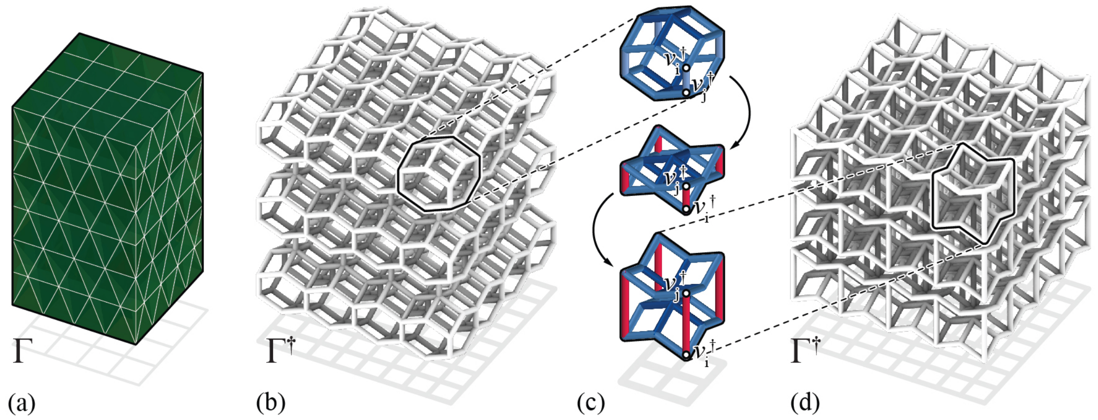
$$\mathbf{M} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}. \quad (2)$$

Matrix **A** contains the reciprocal constraints for the form and force diagram, and matrix **B** represents which edges are constrained. The column vector **t** is obtained by vertically stacking the zero vector and the vector **l**, which encodes the target lengths of the constrained edges. Each solution of **q** represents a set of edge lengths of the form diagram that satisfy all equilibrium requirements and constraints, and it can be used to construct the form diagram.

There are two ways that this algebraic formulation can be used to find a form with concave configurations. First, it can start from a convex configuration and modify the strut lengths based on the GDoF to turn certain cells into concave. Alternatively, the concave configuration can be generated directly from a force diagram with concave cells, which is not supported by the iterative approach as the form diagram can become entangled during the solving process and therefore cannot converge.

2.1. Concave configuration turned from convex configuration through GDoF

When a force diagram only contains convex cells, it usually leads to a form diagram with a convex configuration. However, not all form diagrams can be turned into concave without entangling and self-intersecting. For those that can be turned, there is a common characteristic that they all contain a group of edges with common directions. By assigning proper negative edge lengths to those edges, each cell can be turned concave without creating any self-intersection (Figure 1).



(a) (b) (c) (d)

Figure 1. A simple example that shows a geometry with a convex configuration can be turned concave by assigning negative edge lengths to the parallel edges. (a) An input force diagram, (b) a form diagram with a convex configuration, (c) Negative edge lengths are assigned to the edges that share directions, (d) the modified form with a concave configuration.

2.2. Concave configuration through Directly generated from negative cells

Thanks to the flexibility of mathematical abstraction, all solutions that satisfy the reciprocal relationship can be freely accessed without being limited by geometrical restrictions. Therefore, a force diagram with concave cells can be accepted by the algebraic approach without any issue. The resulting form diagram will directly have a concave configuration (Figure 2). In this case, there is no need to go through the edge length modification process as described in Section 2.1.

3. Verification of the auxetic behavior

After having the geometries with concave configuration generated from Section 2, the finite element method (FEM) is used to verify their auxetic behavior. The specimens are modeled with a footprint

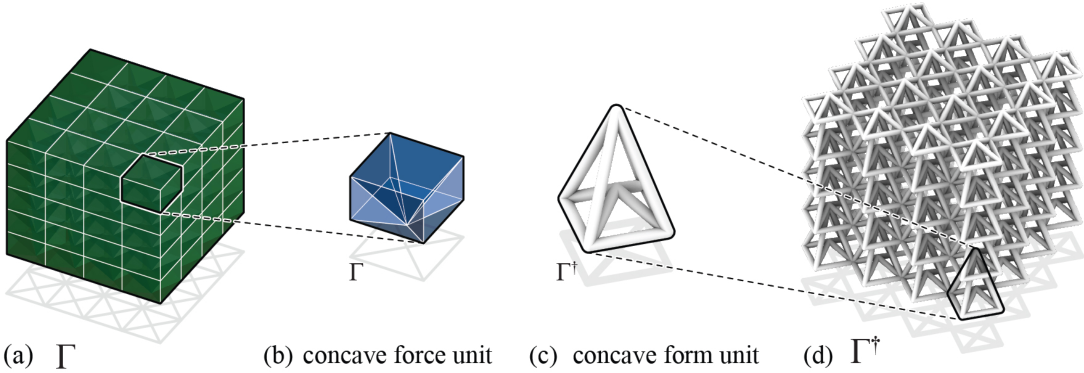


Figure 2. A form geometry with a concave configuration can be generated directly from a force diagram with concave cells. (a) The input force diagram that contains concave cells, (b) a highlighted concave unit of the force diagram, (c) a highlighted concave unit of the form, (d) The form geometry with a concave configuration

of 50 mm by 50 mm, and all edges are materialized as 1.2 mm diameter struts using a linear elastic material. The bottom vertices are fixed in z-direction, and a total vertical force of 10 kN is distributed to all top vertices. The simulation results are shown in Figure 3. The deformation clearly shows those two specimens manifest negative Poisson's ration under vertical loads.

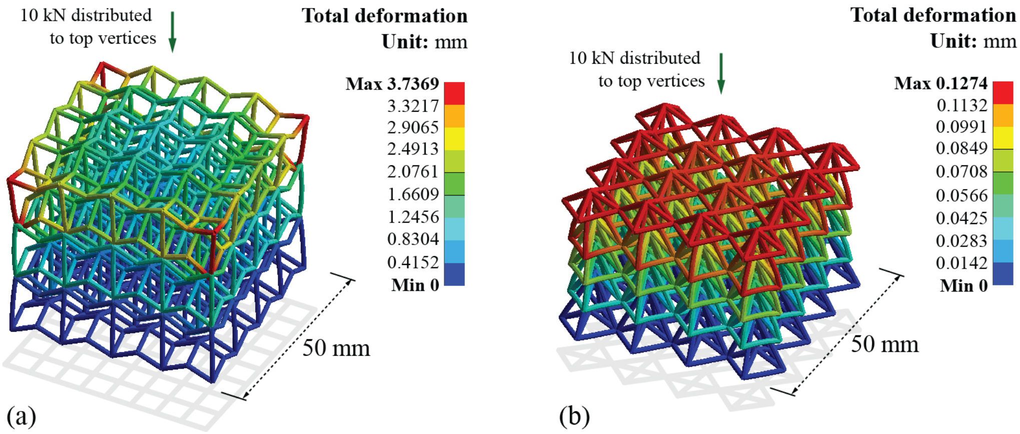


Figure 3. The auxetic behaviors of the two form geometries are verified using FEM. The displacements clearly show negative Poisson's ratios.

The Finite Element Method (FEM) used to simulate the mechanical behavior of materials is computationally intensive and can be time-consuming to set up and run. As an alternative, matrix kinematic analysis can be used to quickly assess the kinematic response to applied loads [30]. This analysis method treats the geometry as a pin-jointed, inextensible framework and ignores its material properties, focusing only on the kinematic behaviors. This lightweight analysis method allows for qualitative and rapid evaluations, making it suitable for fast design iterations and explorations.

4. Conclusion

This paper presents a novel approach to designing 3D auxetic metamaterials using an algebraic formulation of PGS, resulting in concave geometric configurations that are shared by many auxetic materials. The proposed method offers two approaches for generating these concave configurations. The first approach involves starting with a convex polyhedral framework and modifying it using geometric transformations to create a concave configuration. The second approach involves constructing a concave polyhedral framework directly using the algebraic formulation of PGS. Two models are tested using the finite element method, and the results clearly show a negative Poisson's ratio, proving the validity of the proposed method.

By expanding the current design toolset with this novel systematic approach, this research has the potential to drive innovation in various industries. Moreover, this research opens up new possibilities for creating materials with unique mechanical properties that can be tailored to specific applications. This has the potential to revolutionize various fields of engineering, from materials science to civil and mechanical engineering, by enabling the development of materials with unprecedented mechanical properties and performance.

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